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Browder–Weyl theorems, tensor products and multiplications

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ABSTRACT

A Banach space operator $T \in B(\mathcal{X})$ satisfies Browder's theorem if the complement of the Weyl spectrum $\sigma_w(T)$ of T in $\sigma(T)$ equals the set of Riesz points of T; T is polaroid if the isolated points of $\sigma(T)$ are poles (no restriction on rank) of the resolvent of T. Let $\Phi(T)$ denote the set of Fredholm points of T. Browder's theorem transfers from $A, B \in B(\mathcal{X})$ to $S = L_A R_B$ (resp., $S = A \otimes B$) if and only if A and B^* (resp., A and B) have SVEP at points $\mu \in \Phi(A)$ and $\nu \in \Phi(B)$ for which $\lambda = \mu\nu \notin \sigma_w(S)$. If A and B are finitely polaroid, then the polaroid property transfers from $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ to $L_A R_B$; again, restricting ourselves to the completion of $\mathcal{X} \otimes \mathcal{Y}$ in the projective topology, if A and B are finitely polaroid, then the polaroid property transfers from $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ to $A \otimes B$.

1. Introduction

A Banach space operator A, $A \in B(\mathcal{X})$, is polaroid if the isolated points λ of the spectrum of A, $\lambda \in \operatorname{iso}\sigma(A)$, are poles (no restriction on the rank) of the resolvent of A [7]. A necessary and sufficient condition for A to be polaroid is that $\operatorname{asc}(A-\lambda) = \operatorname{dsc}(A-\lambda) < \infty$ for every $\lambda \in \operatorname{iso}\sigma(A)$, where $A - \lambda = A - \lambda I$, $\operatorname{asc}(A-\lambda) =$ the least non-negative integer p such that $(A-\lambda)^{-p}(0) = (A-\lambda)^{-(p+1)}(0)$ is the *ascent* and $\operatorname{dsc}(A-\lambda) =$ the least non-negative integer p such that $(A-\lambda)^p\mathcal{X} = (A-\lambda)^{p+1}\mathcal{X}$ is the *descent* of $A-\lambda$. We say that an operator $A \in B(\mathcal{X})$ is *finitely polaroid*, denoted $A \in p_0(\mathcal{X})$, if the isolated points of $\sigma(A)$ are finite rank poles of the resolvent of A. Recall that an operator A satisfies Weyl's theorem (or, condition) if $\sigma(A) \setminus \sigma_w(A) = \pi_{00}(A)$, and A satisfies Browder's theorem (or, condition) if $\sigma(A) \setminus \sigma_w(A) = \pi_0(A)$ (equivalently, $\sigma_w(T) = \sigma_b(T)$). Here $\sigma_w(A) = \{\lambda \in \mathbb{C}\colon\ A - \lambda$ is not Fredholm or $\operatorname{ind}(T-\lambda) \neq 0\}$ is the Weyl spectrum of A, $\sigma_b(A) = \{\lambda \in \mathbb{C}\colon\ A - \lambda$ is not Fredholm of finite ascent and descent$\}$, $\pi_{00}(A)$ is the set of isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity, $\pi_0(A)$ is the set of Riesz points of A, and $\operatorname{ind}(A)$ is the *index* of A. The following implications hold:

$$\begin{aligned} A \text{ satisfies Weyl's theorem} \quad &\Longrightarrow \quad A \text{ satisfies Browder's theorem} \\ &\Longleftrightarrow \quad A^* \text{ satisfies Browder's theorem.} \end{aligned}$$

Polaroid operators satisfying Weyl's theorem (or, condition) have recently been considered in [3,6,7]. Observe that the polaroid condition is neither necessary nor sufficient for an operator to satisfy Weyl's (even, Browder's) theorem. Consider, for example, the operator $A = R \oplus L \in \ell^p(\mathbb{N})$, where R and L are, respectively, the right shift and the left shift, which is (vacuously) polaroid but does not satisfy Browder's theorem; again, if A is an injective quasi-nilpotent, then A satisfies Weyl's theorem, but is not polaroid. A necessary and sufficient condition for A to satisfy Browder's theorem is that A has the *single-valued extension property*, SVEP, at points $\lambda \notin \sigma_w(A)$ (see [5, Lemma 2.18]). It is straightforward to see [6, Proposition 2.1] that for polaroid operators A with SVEP at points $\lambda \notin \sigma_w(A)$, both A and the conjugate operator A^* satisfy Weyl's theorem.

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For $A, B \in B(\mathcal{X})$, if we let S denote either of the operators $L_A R_B$ and $A \otimes B$, then

$$\sigma_b(S) = \sigma_b(A)\sigma(B) \cup \sigma(A)\sigma_b(B)$$

and

$$\sigma_w(S) \subseteq \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B).$$

Recall from [13] that $\sigma_w(A \otimes B) = \sigma_1 \cup \sigma_2$, where $\sigma_1 = \sigma_e(A)\sigma(B) \cup \sigma(A)\sigma_e(B)$ and σ_2 is the set of all non-zero λ in $\{\sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)\} \setminus \sigma_1$ for which $\operatorname{ind}\{(A \otimes B) - \lambda(I \otimes I)\}$ is non-zero; recall also [10] that if $S = L_A R_B$, then $\sigma_w(S) \subseteq \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$. For operators A and B such that $\sigma_w(S) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$, Browder's theorem transfers from A and B to S (for the simple reason that in such a case $\sigma_b(A) = \sigma_w(A)$ and $\sigma_b(B) = \sigma_w(B)$, and hence that $\sigma_b(S) = \sigma_w(S)$). However, as pointed out by Robin Harte (at KOTAC, Korean Operator Theory and Applications Annual Conference, Seoul, July 2005; see also [11]), it is not apparent that in general Browder's theorem transfers from A and B to S. An operator A is said to be *isoloid* if the isolated points of $\sigma(A)$ are eigenvalues (no restriction on multiplicity) of A. Assuming A and B to be isoloid, Song and Kim [18, Theorem 1] prove that Weyl's theorem transfers from A and B to $A \otimes B$. Their proof, however, depends upon the (liable to fail in general) equality $\sigma_w(A \otimes B) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$.

The isoloid property transfers from operators $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ to tensor products $A \otimes B : \sum_j x_j \otimes y_j \mapsto \sum_j Ax_j \otimes By_j$ $(\mathcal{X} \otimes \mathcal{Y} \to \mathcal{X} \otimes \mathcal{Y})$ and multiplications $L_A R_B : X \mapsto AXB$ $(X \in B(\mathcal{Y}, \mathcal{X}))$ [11]. The polaroid property also transfers successfully to multiplication operators from finitely polaroid operators; moreover, if $\mathcal{X} \otimes \mathcal{Y}$ is complete in the projective tensor product topology, then the polaroid property transfers from finitely polaroid A and B to $A \otimes B$. We also consider the transference of the Browder theorem (the Weyl theorem) from $A, B \in B(\mathcal{X})$ to S. Let $\Phi(T) = \{\lambda \in \mathbb{C}\colon (T - \lambda)\mathcal{X}$ is closed and the deficiency indices $\alpha(T - \lambda) = \dim(T - \lambda)^{-1}(0)$ and $\beta(T - \lambda) = \dim(\mathcal{X}/(T - \lambda)\mathcal{X})$ are finite$\}$ denote the set of Fredholm points of $T \in B(\mathcal{X})$, and let $\sigma_e(T) = \{\lambda \in \mathbb{C}\colon \lambda \notin \Phi(T)\}$ denote the (Fredholm) essential spectrum of T. We prove: Browder's theorem transfers from $A, B \in B(\mathcal{X})$ to $S = L_A R_B$ (resp., $S = A \otimes B$) if and only if A and B^* (resp., A and B) have SVEP at points $\mu \in \Phi(A)$ and $\nu \in \Phi(B)$ for which $\mu\nu = \lambda \notin \sigma_w(S)$. As a consequence we obtain that if A and B are polaroid, A has SVEP at points $\lambda \notin \sigma_e(A)$ and B^* has SVEP at points $\lambda \notin \sigma_e(B)$ (resp., A has SVEP at points $\lambda \notin \sigma_e(A)$ and B has SVEP at points $\lambda \notin \sigma_e(B)$), then $S = L_A R_B$ (resp., $S = A \otimes B$) satisfies Weyl's theorem. It is seen that the Browder and Weyl theorems transfer for a large number of classes of operators.

Except where otherwise stated, we assume in the following that the tensor product $\mathcal{X} \otimes \mathcal{Y}$ is complete with respect to some "suitable cross norm", and that the (left-right multiplication composition) operator $L_A R_B$ acts on the space of operators from $\mathcal{Y}$ to $\mathcal{X}$ or (more generally) one of the Schatten ideals. *Let S denote either of the operators $A \otimes B$ and $L_A R_B$.* Then $\sigma(S) = \sigma(A)\sigma(B)$. An operator $A \in B(\mathcal{X})$ has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to \mathcal{X}$ which satisfies

$$(A - \lambda) f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{D}_{\lambda_0},$$

is the function $f \equiv 0$. Trivially, every operator A has SVEP at points of the resolvent $\rho(A) = \mathbb{C} \setminus \sigma(A)$; also A has SVEP at points $\lambda \in \operatorname{iso}\sigma(A)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.*

It is my pleasure to thank the referee for his suggestions, in particular for bringing reference [9] to my attention.

2. Results

Recall that a necessary and sufficient condition for an operator $A \in B(\mathcal{X})$ to satisfy Browder's theorem is that A has SVEP at points $\lambda \notin \sigma_w(A)$ [5, Lemma 2.18]. Requiring a bit more of the operators $A, B \in B(\mathcal{X})$, we prove that a similar condition is necessary and sufficient for the transference of the Browder condition from A and B to S. But before that we introduce some notation and terminology.

Let $\pi(T) = \{\lambda \in \sigma(T)\colon \operatorname{asc}(T - \lambda) = \operatorname{dsc}(T - \lambda) < \infty\}$ denote the set of poles of the resolvent of the operator T. The *quasi-nilpotent part* $H_0(T - \lambda)$ of $(T - \lambda)$ is defined by

$$H_0(T - \lambda) = \left\{x \in \mathcal{X}\colon \lim_{n \longrightarrow \infty} \left\|(T - \lambda)^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

Let $A, B \in B(\mathcal{X})$. Then $\sigma_e(S) = \sigma_e(A)\sigma(B) \cup \sigma(A)\sigma_e(B)$: this is proved for $S = A \otimes B$ in [13, Theorem 4.2(c)], for $S = L_A R_B$ with A, B Hilbert space operators in [10, Theorem 3.1], and for $S = L_A R_B$ with A, B Banach space operators in [16, Theorem 3.13]. If $\lambda \in \sigma(S) \setminus \sigma_e(S)$, then $\lambda \neq 0$ and the set $E = \{(\mu, \nu) \in \sigma(A)\sigma(B)\colon \mu\nu = \lambda\}$ is finite; this follows from the argument of the proof of [10, Lemma 3.7]. In the case in which $S = L_A R_B$, there exist integers p and n, $p > 0$ and $p \geqslant n \geqslant 0$, distinct non-zero points $\mu_1, \ldots, \mu_p \in \sigma(A) \setminus \sigma_e(A)$ and distinct non-zero points $\nu_1, \ldots, \nu_p \in \sigma(B) \setminus \sigma_e(B)$ such that

(i) $E = \{(\mu_i, \nu_i)\}_{i=1}^{p}$;
(ii) if $n \geqslant 1$, then $\mu_i \in \operatorname{iso}\sigma(A)$ $(1 \leqslant i \leqslant n)$;
(iii) if $p > n$, then $\nu_i \in \operatorname{iso}\sigma(B)$ $(n + 1 \leqslant i \leqslant p)$, and
(iv) $\operatorname{ind}(S - \lambda) = \sum_{j=n+1}^{p} \operatorname{ind}(A - \mu_j)m(B, \nu_j) - \sum_{i=1}^{n} \operatorname{ind}(B - \nu_i)m(A, \mu_i)$,

where $m(T,\alpha) < \infty$ denotes $\dim(H_0(T-\alpha))$ corresponding to the point $\alpha \in \operatorname{iso}\sigma(T)$ [10, Theorem 3.11]. An index formula similar to (iv) (with the minus sign between the sums replaced by the plus sign) holds for tensor products $A \otimes B$ at points $\lambda \neq 0$ [13, Theorem 4.2(e)]. (Another proof of properties (i)–(iv) for $A \otimes B$ and $L_A R_B$ is to be found in [9].)

Theorem 2.1. *Browder's theorem transfers from $A, B \in B(\mathcal{X})$ to $S = L_A R_B$ (resp., $S = A \otimes B$) if and only if A and B^* (resp., A and B) have SVEP at points $\mu \in \Phi(A)$ and $\nu \in \Phi(B)$ for which $\lambda = \mu\nu \notin \sigma_w(S)$.*

Proof. For the sufficiency, we prove that if $\sigma_b(A) = \sigma_w(A)$ and $\sigma_b(B) = \sigma_w(B)$, then $\sigma_b(S) = \sigma_w(S)$. Start by observing that $\sigma_w(S) \subseteq \sigma_b(S) = \sigma_b(A)\sigma(B) \cup \sigma(A)\sigma_b(B) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$. To prove the reverse inclusion, choose a $\lambda \notin \sigma_w(S)$. Then $S - \lambda$ is Fredholm and has index 0, which implies that $A - \mu_i$ and $B - \nu_i$ are Fredholm for $(\mu_i, \nu_i) \in E$. Furthermore,

$$\sum_{j=n+1}^{p} \operatorname{ind}(A-\mu_j)m(B,\nu_j) = \sum_{i=1}^{n} \operatorname{ind}(B-\nu_i)m(A,\mu_i)$$

in the case in which $S = L_A R_B$ and

$$\sum_{j=n+1}^{p} \operatorname{ind}(A-\mu_j)m(B,\nu_j) = -\sum_{i=1}^{n} \operatorname{ind}(B-\nu_i)m(A,\mu_i)$$

in the case in which $S = A \otimes B$. As earlier stated, points $\mu_i \in \operatorname{iso}\sigma(A) \cap \Phi(A)$ and $\nu_j \in \operatorname{iso}\sigma(B) \cap \Phi(B)$; $1 \leqslant i \leqslant n$ and $n+1 \leqslant j \leqslant p$. Hence $\operatorname{ind}(A-\mu_i) = \operatorname{ind}(B-\nu_j) = 0$, and $0 < m(A,\mu_i), m(B,\nu_j) < \infty$, for all $1 \leqslant i \leqslant n$ and $n+1 \leqslant j \leqslant p$ [1, Corollary 3.19]. We claim that $\operatorname{ind}(A-\mu_j) = \operatorname{ind}(B-\nu_i) = 0$ for all $n+1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n$. Let $S = L_A R_B$. Suppose that $\operatorname{ind}(A-\mu_j) \neq 0$ for some j. Then, since $\mu_j \in \Phi(A)$ and A has SVEP at μ_j, $\operatorname{asc}(A-\mu_j) < \infty$ [1, Theorem 3.16] $\Longrightarrow \operatorname{ind}(A-\mu_j) \leqslant 0$ [1, Corollary 3.19]. But then there exists at least one ν_i, $1 \leqslant i \leqslant n$, such that $\operatorname{ind}(B-\nu_i) < 0$. This is not possible, since $\nu_i \in \Phi(B)$ and B^* has SVEP at ν_i imply that $\operatorname{dsc}(B-\nu_i) < \infty$ [1, Theorem 3.17] $\Longrightarrow \operatorname{ind}(B-\nu_i) \geqslant 0$ [1, Corollary 3.19]. Evidently, a similar argument works for the case in which $S = A \otimes B$, A has SVEP at points $\mu \in \Phi(A)$ and B has SVEP at points $\nu \in \Phi(B)$. This proves our claim, and we conclude that $\mu_j \notin \sigma_w(A) = \sigma_b(A)$ and $\nu_i \notin \sigma_w(B) = \sigma_b(B)$ for all $1 \leqslant i, j \leqslant p$. Hence $\lambda \notin \sigma_b(S) \Longrightarrow \sigma_b(S) \subseteq \sigma_w(S)$.

To prove the necessity, we start by observing that $\sigma_w(S) = \sigma_b(S) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$. Let $\mu_i \in \Phi(A)$ and $\nu_i \in \Phi(B)$ be such that $\lambda = \mu_i\nu_i \notin \sigma_w(S)$. Then $1 \leqslant i \leqslant p$ for some integer p, $\mu_i \in \operatorname{iso}\sigma(A)$ $(1 \leqslant i \leqslant n)$ and $\nu_i \in \operatorname{iso}\sigma(B)$ $(n+1 \leqslant i \leqslant p)$ for some integer $n < p$ (see above). If A does not have SVEP at a point μ_j, $n+1 \leqslant j \leqslant p$, then $\operatorname{asc}(A-\mu_j) = \infty$, which implies that $\mu_j \in \sigma_w(A) \Longrightarrow \lambda \in \sigma_w(S)$. Hence A has SVEP at μ_j for all $1 \leqslant j \leqslant n$. Again, if B^* (resp., B) does not have SVEP at a point ν_i, $n+1 \leqslant i \leqslant p$, then $\operatorname{asc}(B^*-\nu_i) = \infty$ (resp., $\operatorname{asc}(B-\nu_i) = \infty$), which implies that $\nu_i \in \sigma_w(B^*) = \sigma_w(B) \Longrightarrow \lambda \in \sigma_w(S)$. $\square$

Remark 2.2. Evidently, a necessary and sufficient condition for $\sigma_w(S) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$ is that $\sigma_w(A)\sigma(B) \subseteq \sigma_w(S)$ and $\sigma(A)\sigma_w(B) \subseteq \sigma_w(S)$. Let (the operators) A, B, S and the points μ_j $(n+1 \leqslant j \leqslant p)$, ν_i $(1 \leqslant i \leqslant n)$ and λ $(= \mu_i\nu_i,\ 1 \leqslant i \leqslant p)$ be as in the proof of Theorem 2.1. Then, for $C = A$ or B and $t = \mu_j$ or ν_i, we have the following possibilities:

(i) $\operatorname{ind}(C-t) = 0$ and $\operatorname{asc}(C-t) = \operatorname{dsc}(C-t) < \infty$;
(ii) $\operatorname{ind}(C-t) < 0$ and $\operatorname{asc}(C-t) < \infty$, $\operatorname{dsc}(C-t) = \infty$ or $\operatorname{asc}(C-t) = \operatorname{dsc}(C-t) = \infty$;
(iii) $\operatorname{ind}(C-t) > 0$ and $\operatorname{asc}(C-t) = \infty$, $\operatorname{dsc}(C-t) < \infty$ or $\operatorname{asc}(C-t) = \operatorname{dsc}(C-t) = \infty$.

(See [12, Theorem 51.1]: observe that the possibility $\operatorname{ind}(C-t) = 0$ and $\operatorname{asc}(C-t) = \operatorname{dsc}(C-t) = \infty$ cannot occur, for the reason that $\sigma_b(C) = \sigma_w(C)$.) If either of the possibilities (ii) and (iii) occurs, then $t \in \sigma_b(C) \Longrightarrow \lambda \in \sigma_b(S)$. The hypothesis A and B (or A and B^*) have SVEP at points in $\Phi(A)$ and $\Phi(B)$, respectively, ensures that only possibility (i) can occur.

The following theorem proves that a version of the polaroid property transfers from A and B to S. But before that some terminology. The *analytic core* $K(A-\lambda)$ of $(A-\lambda)$ is defined by

$$K(A-\lambda) = \big\{x \in \mathcal{X}\colon \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which}$$
$$x = x_0,\ (A-\lambda)(x_{n+1}) = x_n \text{ and } \|x_n\| \leqslant \delta^n\|x\| \text{ for all } n = 1, 2, \ldots\big\}.$$

We note that $H_0(A-\lambda)$ and $K(A-\lambda)$ are (generally) non-closed hyperinvariant subspaces of $(A-\lambda)$ such that $(A-\lambda)^{-q}(0) \subseteq H_0(A-\lambda)$ for all $q = 0, 1, 2, \ldots$ and $(A-\lambda)K(A-\lambda) = K(A-\lambda)$; furthermore, $\mathcal{X} = H_0(T-\lambda) \oplus K(T-\lambda)$ at every $\lambda \in \operatorname{iso}\sigma(T)$ [15].

We shall assume the projective tensor product topology for the space $\mathcal{X} \otimes \mathcal{Y}$ in the case in which $S = A \otimes B$ (and, only in this case) in the following theorem (and its corollaries, namely Corollaries 2.4 and 2.5).

Theorem 2.3. *A and B finitely polaroid implies S polaroid.*

Proof. Suppose that $A \in p_0(\mathcal{X})$ and $B \in p_0(\mathcal{Y})$ and that $\lambda \in \operatorname{iso}\sigma(S)$. We divide the proof into the cases $\lambda \neq 0$ and $\lambda = 0$.

Case: $\lambda \neq 0$. Let $\lambda = \mu\nu$, where $\mu \in \operatorname{iso}\sigma(A)$ and $\nu \in \operatorname{iso}\sigma(B)$. The hypotheses $A \in p_0(\mathcal{X})$ and $\mu \in \operatorname{iso}\sigma(A)$ imply that

$$\mathcal{X} = H_0(A-\mu) \oplus K(A-\mu) = (A-\mu)^{-q}(0) \oplus (A-\mu)^q \mathcal{X},$$

for some positive integer q, where $\dim(H_0(A-\mu)) = \alpha(A-\mu) < \infty$. Recall from [14, Proposition 3.7.5] that a point $\mu \in \operatorname{iso}\sigma(A)$ belongs to $\sigma_e(A)$ if and only if $\dim(H_0(A-\mu))$ is infinite. Hence $\mu \notin \sigma_e(A)$, and, by a similar argument, $\nu \notin \sigma_e(B)$. Since $\sigma_e(S) = \sigma_e(A)\sigma(B) \cup \sigma(A)\sigma_e(B)$, it follows that $\lambda \notin \sigma_e(S)$, i.e., $\lambda \in \Phi(S)$. The point λ being isolated in $\sigma(S)$, both S and S^* have SVEP at λ. Applying [1, Corollary 3.21] it follows that λ is a pole of the resolvent of S.

Case: $\lambda = 0$. The case $\lambda = 0$ differs from the case $\lambda \neq 0$ in as much as that we require only that A and B are polaroid. Observe that if $0 \in \operatorname{iso}\sigma(S)$, then we have the following cases: (i) 0 is an isolated point of both $\sigma(A)$ and $\sigma(B)$; (ii) 0 is an isolated point of $\sigma(A)$ (or $\sigma(B)$) and $0 \notin \sigma(B)$ (resp., $0 \notin \sigma(A)$); (iii) 0 is an accumulation point of $\sigma(A)$ (or, $\sigma(B)$) and $\sigma(B) = \{0\}$ (resp., $\sigma(A) = \{0\}$). Clearly, if case (iii) holds with $0 \in \operatorname{acc}\sigma(A)$ and $\sigma(B) = \{0\}$, then B is a nilpotent operator, which implies that S is a nilpotent operator (which in turn implies that 0 is a pole). We prove next that if either of the cases (i) and (ii) holds, then S has finite descent. Since S has SVEP at 0, this would then imply that $\operatorname{asc}(S) = \operatorname{dsc}(S) < \infty$ [1, Theorem 3.81]. If $0 \in \operatorname{iso}\sigma(A)$ and $0 \in \operatorname{iso}\sigma(B)$, then $\operatorname{asc}(A) = \operatorname{dsc}(A) \leqslant q_1 < \infty$ and $\operatorname{asc}(B) = \operatorname{dsc}(B) \leqslant q_2 < \infty$ for some positive integers q_1 and q_2. Let $q = \max(q_1, q_2)$. Then the mappings $A : A^q\mathcal{X} \to A^q\mathcal{X}$ and $B : B^q\mathcal{Y} \to B^q\mathcal{Y}$ are bijective. Hence the induced mapping $A \otimes B : A^q\mathcal{X} \otimes B^q\mathcal{Y} \to A^q\mathcal{X} \times B^q\mathcal{Y}$ is also bijective. (This is where we avail ourselves of the completion of $\mathcal{X} \otimes \mathcal{Y}$ in the projective tensor product topology.) Thus

$$S^{q+1}(\mathcal{X} \otimes \mathcal{Y}) = (A \otimes B)\big(A^q\mathcal{X} \otimes B^q\mathcal{Y}\big) = S^q(\mathcal{X} \otimes \mathcal{Y}).$$

Now let $S = L_A R_B$. The operators A and B being polaroid, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where $A_1 = A|_{\mathcal{X}_1}, B_1 = B|_{\mathcal{Y}_1}$ are q-nilpotent and $A_2 = A|_{\mathcal{X}_2}, B_2 = B|_{\mathcal{Y}_2}$ are surjective. Suppose that $Y = A^{q+1}XB^{q+1}$ for some $X \in B(\mathcal{Y}, \mathcal{X})$. Letting $X : \mathcal{Y}_1 \oplus \mathcal{Y}_2 \to \mathcal{X}_1 \oplus \mathcal{X}_2$ have the matrix representation $X = [X_{ij}]_{i,j\leqslant 1}^2$, it then follows that $Y = 0 \oplus A_2^{q+1}X_{22}B_2^{q+1} = A^qZB^q$, where $Z = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & A_2X_{22}B_2 \end{pmatrix}$. Hence

$$\begin{aligned} Y \in \operatorname{ran}(L_AR_B)^{q+1} \quad &\Longrightarrow \quad Y = L_A^{q+1}XR_B^{q+1} \quad \text{for some } X \in B(\mathcal{Y}, \mathcal{X}) \\ &= \quad L_A^q Z R_B^q \quad \text{for some } Z \in B(\mathcal{Y}, \mathcal{X}) \text{ such that } Z \notin (L_AR_B)^{-q}(0) \\ &\Longrightarrow \quad Y \in \operatorname{ran}(L_AR_B)^q. \end{aligned}$$

Since the same argument works with $q = q_1$ in the case in which $0 \in \operatorname{iso}\sigma(A)$ and $0 \notin \sigma(B)$, the proof is complete. □

The polaroid condition does not reverse: consider, for example, the operators $A = I \oplus Q$, where Q is a (non-nilpotent) quasi-nilpotent, and $B =$ the forward unilateral shift, when it is seen that S is (vacuously) polaroid but A is not polaroid. The following corollary relates points $\lambda \in \pi_{00}(S)$ to points in $\pi_0(S)$.

Corollary 2.4. *If A and B are finitely polaroid, then* $\lambda \in \pi_{00}(S) \Longleftrightarrow \lambda \in \pi_0(S)$.

Proof. Evidently, $\pi_0(S) \subseteq \pi_{00}(S)$. Let $\lambda \in \pi_{00}(S)$. Then $\lambda \notin \sigma_e(S)$, for the reason that if $\lambda \in \sigma_e(S) \cap \operatorname{iso}\sigma(S)$, then $\dim(H_0(S-\lambda)) = \alpha(S-\lambda) = \infty$. Since S is polaroid at points $\lambda \in \operatorname{iso}\sigma(S) \setminus \sigma_e(S)$, $\lambda \in \pi_0(S) \Longrightarrow \pi_{00}(S) = \pi_0(S)$. □

Corollary 2.5. *If $A \in B(\mathcal{X})$ has SVEP at points $\mu \in \Phi(A)$, $B \in B(\mathcal{X})$ has SVEP at points $\nu \in \Phi(B)$ (resp., $A \in B(\mathcal{X})$ has SVEP at points $\mu \in \Phi(A)$ and $B^* \in B(\mathcal{X}^*)$ has SVEP at points $\nu \in \Phi(B)$), and if A and B are finitely polaroid, then $A \otimes B$ and $(A \otimes B)^*$ (resp., L_AR_B and $(L_AR_B)^*$) satisfy Weyl's theorem.*

Proof. Evidently, A, B and S satisfy Browder's theorem; in particular, $\sigma(S) \setminus \sigma_w(S) = \sigma(S^*) \setminus \sigma_w(S^*) = \pi_0(S) = \pi_0(S^*)$. Hence $\sigma(S) \setminus \sigma_w(S) \subseteq \pi_{00}(S)$ and $\sigma(S^*) \setminus \sigma_w(S^*) \subseteq \pi_{00}(S^*)$. If $\lambda \in \pi_{00}(S)$ (or $\pi_{00}(S^*)$), then (see Corollary 2.4) $\lambda \in \operatorname{iso}\sigma(S) \Longrightarrow \lambda \in \pi_0(S) = \pi_0(S^*)$. Hence $\sigma(S) \setminus \sigma_w(S) = \sigma(S^*) \setminus \sigma_w(S^*) = \pi_{00}(S) = \pi_{00}(S^*)$. □

The finite polaroid hypothesis of Corollary 2.5 can be relaxed as follows.

Theorem 2.6. *Suppose that $A, B \in B(\mathcal{X})$ are polaroid. If A has SVEP at points $\mu \in \Phi(A)$, B (resp., B^*) has SVEP at points $\nu \in \Phi(B)$ in the case in which $S = A \otimes B$ (resp., $S = L_AR_B$), then S and S^* satisfy Weyl's theorem.*

Proof. Recall that a (necessary and) sufficient condition for an operator T to satisfy Browder's theorem is that T has SVEP at points $\lambda \notin \sigma_w(T)$ [5]. Thus, since $\sigma_e(T) \subseteq \sigma_w(T)$ for every operator T, the hypotheses A has SVEP at points $\mu \in \Phi(A)$ and B has SVEP at points $\nu \in \Phi(B)$ imply that both A and B satisfy Browder's theorem. Again, the hypothesis B^* has SVEP at points $\nu \in \Phi(B)$ $(= \Phi(B^*))$ implies that B^* satisfies Browder's theorem; since B satisfies Browder's theorem if and only if B^* satisfies Browder's theorem, a consequence of the fact that $\sigma_w(B) = \sigma_w(B^*)$ and $\sigma_b(B) = \sigma_b(B^*)$, once again B satisfies Browder's theorem. Hence A, A^*, B and B^* (all) satisfy Browder's theorem. Apparently, the hypothesis that A and B are polaroid implies A^* and B^* are polaroid. Since a polaroid operator satisfying Browder's theorem satisfies Weyl's theorem [6, Theorem 2.2(i), (ii)], the operators A, A^*, B and B^* satisfy Weyl's theorem. In particular, $\sigma_w(C) \cap \pi_{00}(C) = \emptyset$, where C stands for either of the operators A, A^*, B and B^*. Evidently, polaroid operators are isoloid. Applying [11, Theorems 7 and 8], it follows that $\sigma_w(D) \cap \pi_{00}(D) = \emptyset$, where D stands for either of S and S^*. Since S satisfies Browder's theorem ($\Longleftrightarrow$ S^* satisfies Browder's theorem) by Theorem 2.1, D satisfies Weyl's theorem. □

Let $\mathcal{H}(S)$ denote the set of non-constant functions f which are (defined and) analytic on an open neighborhood of $\sigma(S)$.

Corollary 2.7. *If $A, B \in B(\mathcal{X})$ satisfy the hypotheses of Theorem* 2.6*, then $f(S)$ and $f(S^*)$ satisfy Weyl's theorem for every $f \in \mathcal{H}(S)$.*

Proof. Evidently, S and S^* satisfy Weyl's theorem. Since the SVEP hypotheses on A, B when $S = A \otimes B$, and on A, B^* when $S = L_A R_B$, imply that $\operatorname{ind}(S - \lambda) \leqslant 0$ for all $\lambda \in \Phi(S)$, $f(S)$ and $f(S^*) = f(S)^*$ also satisfy Weyl's theorem [17, Theorem 1]. □

Examples. If T is a *decomposable operator*, then (both) T and T^* have SVEP [1, Theorem 6.32]; hence for decomposable operators A and $B \in B(\mathcal{X})$, $L_A R_B$ and $A \otimes B$ satisfy Browder's theorem. An interesting class of operators with SVEP is the class $H(p)$ of operators T which satisfy the property that

$$H_0(T - \lambda) = (T - \lambda)^{-p}(0)$$

for some integer $p \geqslant 1$ and all complex λ. Class $H(p)$ is large: it contains (amongst others) the class consisting of *generalized scalar*, *subscalar* and *totally paranormal operators* on a Banach space, *hyponormal* ($|T^*|^2 \leqslant |T|^2$), *multipliers of semi-simple Banach algebras*, and *p-hyponormal* ($|T^*|^{2p} \leqslant |T|^{2p}$ for some $0 < p < 1$), *M-hyponormal* (there exists a scalar $M \geqslant 1$ such that $|(T-\lambda)^*|^2 \leqslant M|T-\lambda|^2$ for all complex λ) and *totally $*$-paranormal operators* ($\|(T-\lambda)^* x\|^2 \leqslant \|(T-\lambda)^2 x\|$ for every unit vector x) on a Hilbert space (see [1, Section 3.8]). Evidently, $L_A R_{B^*}$ and $A \otimes B$ satisfy Browder's theorem for $A, B \in H(p)$. More is true.

Theorem 2.8. *If $A, B \in H(p) \cap B(\mathcal{X})$, then $f(S)$ and $f(S^*)$, $S = L_A R_{B^*}$ or $A \otimes B$, satisfy Weyl's theorem for every $f \in \mathcal{H}(S)$.*

Proof. It is easily seen that operators $T \in H(p)$ have finite ascent ($\Longrightarrow$ SVEP) and are polaroid. Apply Theorem 2.6 and Corollary 2.7. □

Another important class of operators whose elements T have SVEP at points $\lambda \in \Phi(T)$ is the class $\mathcal{CHN}$ of *completely hereditarily normaloid operators*, defined as follows. A part of an operator is its restriction to a closed invariant subspace. An operator $T \in B(\mathcal{X})$ is a $\mathcal{CHN}$ operator if either (i) every part of $T - \lambda$ is normaloid for every $\lambda \in \mathbb{C}$, or, (ii) every part of T and the inverse of every invertible part of T is normaloid. The class $\mathcal{CHN}$ is large. In particular, Hilbert space operators T which are either hyponormal or p-hyponormal or *w-hyponormal* (if T has the polar decomposition $T = U|T|$, and $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$, then $|T| \leqslant |\tilde{T}^*| \leqslant |\tilde{T}|$) are $\mathcal{CHN}$ operators. Again, totally $*$-paranormal Hilbert space operators, and *paranormal operators* $T \in B(\mathcal{X})$ ($\|Tx\|^2 \leqslant \|T^2 x\|$ for all unit vectors $x \in \mathcal{X}$) are $\mathcal{CHN}$ operators.

Theorem 2.9. *If $A, B \in \mathcal{CHN}$, then $f(S)$ and $f(S^*)$, $S = L_A R_{B^*}$ or $A \otimes B$, satisfy Weyl's theorem for every $f \in \mathcal{H}(S)$.*

Proof. The isolated points of the spectrum of a $\mathcal{CHN}$-operator are simple poles of the resolvent of the operator [5, Proposition 2.1]. Since for operators $T \in \mathcal{CHN}$ both T and T^* have SVEP at their Fredholm points, this follows from the argument of the proof of Theorem 2.9 of [5], Theorem 2.6 and Corollary 2.7 apply. □

Remark 2.10. Some classes of operators A and B have the property that their tensor product $A \otimes B$ again belongs to the class. Thus, if A and B are hyponormal (even, p-hyponormal) Hilbert space operators, then $A \otimes B$ is hyponormal (resp., p-hyponormal) [4]. Evidently, both the Browder and Weyl theorems transfer from hyponormal (resp., p-hyponormal) A and B to $A \otimes B$. It is known that if A and B^* are hyponormal (Hilbert space) operators, then $H_0(L_A R_B - \lambda) = (L_A R_B - \lambda)^{-1}(0)$ for all complex λ [8, Proof of Lemma 3.5]; hence, once again, Weyl's theorem and Browder's theorem transfer from hyponormal A and B^* to $L_A R_B$. Recall [4], however, that the tensor product of paranormal operators may fail to be paranormal. It is, in view of this, of interest that the Browder and Weyl theorems transfer from paranormal A and B to $A \otimes B$ and $L_A R_{B^*}$.

Remark 2.11. Song and Kim [18, Theorem 1] prove that *if $A, B \in B(\mathcal{X})$ are isoloid and both A, B satisfy Weyl's theorem, then $A \otimes B$ satisfies Weyl's theorem*. Their proof however depends upon the equality $\sigma_w(A \otimes B) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$, an equality which fails to be true in general. Also, as earlier pointed out, it is seemingly not known if $\sigma_w(L_A R_B) =$

$\sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$. (It would be of some interest to have an example showing that the inclusion $\sigma_w(L_A R_B) \subset \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$ is proper, though at the moment such an example seems to be elusive.) The results of this paper, particularly Theorem 2.1, have a bearing on the results of [2] (in particular, Lemma 2.5, and consequently Theorem 2.1), which (again) draw upon results from a pre-print dependent upon the assumption that $\sigma_w(S) = \sigma_w(A)\sigma(B) \cup \sigma(A)\sigma_w(B)$.

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